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COMPACT PERTURBATIONS OF CERTAIN VON NEUMANN ALGEBRAS

BY

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ABSTRACT. Let $\mathcal{E}$ be a sequence of mutually orthogonal, finite dimensional projections whose sum is the identity on a Hilbert space $\mathcal{H}$. If we denote the commutant of $\mathcal{E}$ by $\mathfrak{D}(\mathcal{E})$ and the ideal of compact operators on $\mathcal{H}$ by $\mathcal{C}(\mathcal{H})$, then it is easily verified that $\mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H}) = \{T + K: T \in \mathfrak{D}(\mathcal{E}), K \in \mathcal{C}(\mathcal{H})\}$ is a C^*-algebra. In this paper we classify all such algebras up to $*$-isomorphism and characterize them by examining their relationship to certain quasidiagonal and quasitriangular algebras.

Throughout this paper, $\mathcal{E}$ will denote a sequence $\{E_i\}_{i=1}^{\infty}$ of mutually orthogonal, finite dimensional projections on a Hilbert space whose sum is the identity. The set consisting of every bounded operator whose matrix representation is diagonal with respect to $\mathcal{E}$ will be referred to as the *block diagonal algebra associated with* $\mathcal{E}$ and denoted as $\mathfrak{D}(\mathcal{E})$; we remark that $\mathfrak{D}(\mathcal{E})$ can be viewed as the commutant of $\mathcal{E}$, or as the direct product of the von Neumann algebras $\mathcal{L}(E_i\mathcal{H})$, where $\mathcal{L}(\mathcal{H})$ (respectively $C(\mathcal{H})$) will denote the algebra of bounded (respectively compact) operators on a separable Hilbert space.

DEFINITION. $\mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H}) = \{T + K: T \in \mathfrak{D}(\mathcal{E}), K \in \mathcal{C}(\mathcal{H})\}$ will be referred to as the *perturbed block diagonal algebra for* $\mathcal{E}$.

By [2, Chapter 1], $\mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$ is a C^*-algebra. It is weak operator dense in $\mathcal{L}(\mathcal{H})$ because $\mathcal{C}(\mathcal{H})$ is and is not norm separable because $\mathfrak{D}(\mathcal{E})$ is not. It is not hard to realize that the structure of such an algebra depends only on the dimensions of the projections in $\mathcal{E}$, not on how we label them, so we will occasionally suppress $\mathcal{E}$ and wirte $\mathfrak{D}\{(\alpha_i)_{i=1}^{\infty}\} + \mathcal{C}(\mathcal{H})$ instead, where α_i is the dimension of E_i for each i.

One might guess that two perturbed block diagonal algebras are isomorphic if their associated sequences agree after a finite number of terms. Indeed, $\mathfrak{D}\{(2,1,1,1,\ldots.)\} + \mathcal{C}(\mathcal{H})$ is isomorphic to $\mathfrak{D}\{(1,1,1,1,\ldots.)\} + \mathcal{C}(\mathcal{H})$. But

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what is perhaps surprising is that $\mathfrak{D}\{(1,2,2,2,\ldots.)\} + \mathcal{C}(\mathcal{H})$ is not isomorphic to $\mathfrak{D}\{(2,2,2,2,\ldots.)\} + \mathcal{C}(\mathcal{H})$.

THEOREM 1. *Let* $\mathfrak{A} = \mathfrak{D}(\mathfrak{E}) + \mathcal{C}(\mathcal{H})$ *and* $\mathfrak{B} = \mathfrak{D}(\mathfrak{F}) + \mathcal{C}(\mathcal{H})$. $\mathfrak{A}$ *is isomorphic to* $\mathfrak{B}$ *if and only if there exist finite subsets of the positive integers,* N_0 *and* N_1, *and a bijection* $\tau: \mathbf{N} - N_0 \to \mathbf{N} - N_1$ *such that*

$$\dim\left(\sum_{i\in N_0} E_i\right) = \dim\left(\sum_{j\in N_1} F_j\right) \quad \textit{and} \quad \dim E_i = \dim F_{\tau(i)} \quad \textit{for all } i \in \mathbf{N} - N_0.$$

We include a brief discussion of the important ideas in the proof of Theorem 1. That the condition is sufficient to insure isomorphism follows from the construction of a unitary operator U such that

$$U\left(\sum_{i\in N_0} E_i\right)U^* = \sum_{i\in N_1} F_i \quad \text{and} \quad UE_iU^* = F_{\tau(i)} \quad \text{for all } i \text{ in } \mathbf{N} - N_0,$$

and the realization that U implements an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. That the condition is necessary to insure isomorphism is less obvious; the main technical effort is contained in the following lemma:

LEMMA 2. *Any selfadjoint unitary operator U which implements an automorphism of a perturbed block diagonal algebra* $\mathfrak{D}(\mathfrak{E}) + \mathcal{C}(\mathcal{H})$ *is a compact perturbation of a selfadjoint unitary operator V which implements an automorphism of* $\mathfrak{D}(\mathfrak{E})$.

Assuming the validity of Lemma 2, we sketch the remainder of the proof as follows:

PROOF OF THEOREM 1. If $\mathfrak{A}$ and $\mathfrak{B}$ are the isomorphic algebras of Theorem 1, we let

$$C = \left\{\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22}\end{pmatrix}: T_{11} \in \mathfrak{A}, T_{22} \in \mathfrak{B}; T_{21}, T_{12} \in \mathcal{C}(\mathcal{H})\right\}.$$

It is easily verified that $\mathcal{C}$ is a perturbed block diagonal algebra on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$; i.e., $\mathcal{C} = \mathfrak{D}(\mathfrak{E}) \oplus \mathfrak{D}(\mathfrak{F}) + \mathcal{C}(\mathcal{K})$. It follows from [8, Lemma 4.5] that any isomorphism α between $\mathfrak{A}$ and $\mathfrak{B}$ is implemented by a unitary operator W. We then remark that the unitary operator U, whose representation as a matrix on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$ is $\left(\begin{smallmatrix} 0 & W^* \\ W & 0\end{smallmatrix}\right)$, induces an automorphism of $\mathcal{C}$.

After noting that U is selfadjoint, we apply Lemma 2 to produce a selfadjoint operator V which implements an automorphism of $\mathfrak{D}(\mathfrak{E}) \oplus \mathfrak{D}(\mathfrak{F})$ and such that $V = U + Q$, where Q is a compact operator on $\mathcal{K}$ and is selfadjoint because V and U are. Hence,

$$V = U + \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22}\end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} + W^* \\ Q_{21} + W & Q_{22}\end{pmatrix}.$$

Since V implements an automorphism of $\mathfrak{D}(\mathfrak{E}) \oplus \mathfrak{D}(\mathfrak{F})$ it shuffles the projections in $\mathfrak{E} \cup \mathfrak{F}$ and since V is selfadjoint, we conclude that there exist subsets N_0 and N_1 of $\mathbf{N}$ and bijections:

$$\tau_0 \colon N_0 \to N_0 \text{ such that } VE_iV^* = E_{\tau_0(i)} \text{ for all } i \text{ in } N_0,$$

$$\tau_1 \colon N_1 \to N_1 \text{ such that } VF_jV^* = F_{\tau_1(j)} \text{ for all } j \text{ in } N_1, \text{ and}$$

$$\tau \colon \mathbf{N} - N_0 \to \mathbf{N} - N_1 \text{ such that } VE_iV^* = F_{\tau(i)} \text{ for all } i \text{ in } \mathbf{N} - N_0.$$

Since V is unitary, $\dim(E_i) = \dim(VE_iV^*) = \dim(F_{\tau(i)})$ for all $i \in \mathbf{N} - N_0$, so it remains only to show that N_0 is a finite subset of $\mathbf{N}$ and that $\dim(\Sigma_{i \in N_0} E_i) = \dim(\Sigma_{j \in N_1} F_j)$. One can conclude from the matrix representation of V that Q_{11} is a selfadjoint partial isometry and that $\Sigma_{i \in N_0} E_i$ is the projection onto the initial space of Q_{11} while $\Sigma_{j \in N_1} F_j$ is the projection onto the initial space of $Q_{22}(*)$.

Since Q_{11} is compact, we deduce immediately that N_0 is finite. Note, also, that the initial space of Q_{11} is the kernel of $Q_{21} + W$. Since $Q_{21} + W$ is a compact perturbation of a unitary operator, the index of $Q_{21} + W$ is zero, implying that

$$\dim \ker(Q_{21} + W) = \dim \ker(Q_{21} + W)^* = \dim \ker(Q_{12} + W^*),$$

with the last equality following from the fact that Q is selfadjoint. Since the kernel of $Q_{12} + W^*$ is the initial space of Q_{22}, we conclude that the dimension of the initial space of Q_{11} equals the dimension of the initial space of Q_{22}. It follows from $(*)$ that $\dim(\Sigma_{i \in N_0} E_i) = \dim(\Sigma_{j \in N_1} F_j)$, thus concluding the proof of Theorem 1. □

We now proceed (from Proposition 3 through Lemma 9) to gather the facts necessary to verify Lemma 2.

PROPOSITION 3. *Let $\mathfrak{F} = \{F_n\}_{n=1}^{\infty}$ be any sequence of finite dimensional, mutually orthogonal projections. Then $\|KF_n\| \to 0$ for every compact operator K.*

PROOF. Arguing as in [3, Lemma 1, p. 292] this is easily verified for the finite rank operators and because these are norm-dense in $\mathcal{C}(\mathcal{H})$, the assertion follows by an application of the triangle inequality. □

DEFINITION. Let $\delta_{\mathfrak{E}}$ denote the map which takes an operator T to the formal sum $\delta_{\mathfrak{E}}(T) = \Sigma_{k=1}^{\infty} E_k T E_k$, by which we mean the strong limit of $\Sigma_{k=1}^{n} E_k T E_k$. Since

$$\|\delta_{\mathfrak{E}}(T)\| = \left\| \sum_{k=1}^{\infty} E_k T E_k \right\| = \sup_k \|E_k T E_k\| \leq \|T\| \quad \text{and} \quad \|\delta_{\mathfrak{E}}(1)\| = \|1\|,$$

$\delta_{\mathfrak{E}}$ is a linear operator on $\mathcal{L}(\mathcal{H})$ of norm one.

PROPOSITION 4. *Suppose $T \in \mathfrak{D}(\mathfrak{E}) + \mathcal{C}(\mathcal{H})$. Then $T - \delta_{\mathfrak{E}}(T)$ is compact.*

PROOF. From the definition of $\mathfrak{D}(\mathfrak{E}) + \mathcal{C}(\mathcal{H})$, $T = S + K$, where $S \in$

$\mathfrak{D}(\mathfrak{E})$ and $K \in \mathcal{C}(\mathcal{H})$. So, $\delta_{\mathfrak{E}}(T) = \delta_{\mathfrak{E}}(S + K) = S + \delta_{\mathfrak{E}}(K) = T - K + \delta_{\mathfrak{E}}(K)$ and $T - \delta_{\mathfrak{E}}(T) = K - \delta_{\mathfrak{E}}(K)$. It follows that $T - \delta_{\mathfrak{E}}(T)$ is compact because K and $\delta_{\mathfrak{E}}(K)$ are. □

LEMMA 5. *Assume that U is a unitary operator which implements an automorphism of $\mathfrak{D}(\mathfrak{E}) + \mathcal{C}(\mathcal{H})$. Then* $\lim_i \inf_j \|E_j^{\perp} UE_i\| = 0$.

PROOF. In this contrapositive argument we assume that $\inf_j \|E_j^{\perp} UE_i\| \nrightarrow 0$ and then show that U does not implement an automorphism of $\mathfrak{D}(\mathfrak{E}) + \mathcal{C}(\mathcal{H})$. So there exist $\varepsilon > 0$ and an infinite subset M of $\mathbf{N}$ such that $\inf_{j \in \mathbf{N}} \|E_j^{\perp} UE_i\| \geqslant \varepsilon$ for all $i \in M$ (*). We will inductively obtain a set of finite dimensional, mutually orthogonal projections $\{C_{i_k}, D_{i_k}\}_{k=1}^{\infty}$ for which $\|C_{i_k} UE_{i_k}\|$ and $\|D_{i_k} UE_{i_k}\|$ are both greater than $\varepsilon/4$. Let i_1 be the first integer in M. There are two possibilities:

(1) If $\max_j \|E_j UE_{i_1}\| \geqslant \varepsilon/4$ then let j_1 be any integer such that $\|E_{j_1} UE_{i_1}\| \geqslant \varepsilon/4$. By (*) it follows that $\|\Sigma_{j \neq j_1} E_j UE_{i_1}\| \geqslant \varepsilon$ and since $\Sigma_{j=1, j \neq j_1}^{n} E_j UE_{i_1}$ tends strongly to $\Sigma_{j \neq j_1} E_j UE_{i_1}$, it follows that for some $n_1 \in \mathbf{N}$, $\|\Sigma_{j \neq j_1}^{n_1} E_j UE_{i_1}\| \geqslant \varepsilon/4$. Let $C_{i_1} = E_{j_1}$ and $D_{i_1} = \Sigma_{j=1, j \neq j_1}^{n_1} E_j$.

(2) If $\max_j \|E_j UE_{i_1}\| < \varepsilon/4$, let n_1 be the first integer for which $\|\Sigma_{j=1}^{n_1} E_j UE_{i_1}\| \geqslant \varepsilon/4$. Then

$$\left\| \sum_{j=1}^{n_1} E_j UE_{i_1} \right\| < \left\| \sum_{j=1}^{n_1 - 1} E_j UE_{i_1} \right\| + \|E_{j_{n_1}} UE_{i_1}\| < \varepsilon/2.$$

Hence $\|\Sigma_{j=n_1+1}^{\infty} E_j UE_{i_1}\| \geqslant \varepsilon/2$ because $\|UE_{i_1}\| = 1 \geqslant \varepsilon$ and so there exists n_2 such that $\|\Sigma_{j=n_1+1}^{n_2} E_j UE_{i_1}\| \geqslant \varepsilon/4$. Let $C_{i_1} = \Sigma_{j=1}^{n_1} E_j$ and $D_{i_1} = \Sigma_{j=n_1+1}^{n_2} E_j$.

For the kth inductive step, let i_k be the first integer in M such that $\|(\Sigma_{j=1}^{k-1} C_{i_j} + D_{i_j}) UE_{i_k}\| < \varepsilon/8$. That such an integer exists follows from Proposition 3 and the fact that $(\Sigma_{j=1}^{k-1} C_{i_j} + D_{i_j}) U$ is compact. Since $\|(\Sigma_{j=1}^{k-1} C_{i_j} + D_{i_j})^{\perp} UE_{i_k}\| \geqslant 7\varepsilon/8$, the reader can verify that we are able to repeat the process to obtain mutually orthogonal, finite dimensional subprojections of $(\Sigma_{j=1}^{k-1} C_{i_j} + D_{i_j})^{\perp}$, C_{i_k} and D_{i_k}, such that $\|C_{i_k} UE_{i_k}\|$, $\|D_{i_k} UE_{i_k}\| \geqslant \varepsilon/4$.

We now pass to a subsequence $\{h_k\}$ of $\{i_k\}$ such that for every $k \in \mathbf{N}$,

$$\sum_{l \neq k} \|C_{h_l} UE_{h_k}\| \cdot \|D_{h_l} UE_{h_k}\| < \varepsilon^2/32.$$

To do so, let $\{\alpha_{i,j}\}_{i,j \in \mathbf{N}}$ be a set of positive real numbers for which $\Sigma_{i,j \in \mathbf{N}} \alpha_{ij}^2 < \varepsilon^2/32$. Let $h_1 = i_1$. Assuming that we have obtained $\{h_k\}_{k=1}^{n}$, let h_{n+1} be the next integer in $\{i_k\}$ such that for every $l \lneq n + 1$, $\|C_{h_{n+1}} UE_{h_l}\|$ and $\|D_{h_{n+1}} UE_{h_l}\|$ are each less than $\alpha_{n+1,l}$, whereas $\|C_{h_l} UE_{h_{n+1}}\|$ and $\|D_{h_l} UE_{h_{n+1}}\|$ are each less than $\alpha_{l,n+1}$; this is possible by Proposition 3 applied to UE_{h_l}, $C_{h_l} U$, and $D_{h_l} U$ for those $l \lneq n + 1$. Continue inductively.

We will construct an operator $T \in \mathfrak{D}(\mathfrak{E})$ such that $UTU^* \notin \mathfrak{D}(\mathfrak{E}) +$

$\mathcal{C}(\mathcal{H})$, thus concluding the argument. Since $\|C_{h_k}UE_{h_k}\|$ and $\|E_{h_k}U^*D_{h_k}\|$ are both greater than or equal to $\varepsilon/4$, there is a rank one partial isometry $T_{h_k} \in \mathcal{L}(E_{h_k}\mathcal{H})$ for which $\|C_{h_k}UE_{h_k}T_{h_k}E_{h_k}U^*D_{h_k}\| > \varepsilon^2/16$ and we let $T = \Sigma_{k\in\mathbf{N}}T_{h_k}$. For arbitrary $l \in \mathbf{N}$,

$$C_{h_l}UTU^*D_{h_l} = C_{h_l}UT_{h_l}U^*D_{h_l} + \sum_{k\neq l} C_{h_l}UT_{h_k}U^*D_{h_l}.$$

Hence,

$$\begin{aligned}\|C_{h_l}UTU^*D_{h_l}\| &\geq \|C_{h_l}UT_{h_l}U^*D_{h_l}\| - \left\|\sum_{k\neq l} C_{h_l}UT_{h_k}U^*D_{h_l}\right\| \\ &\geq \|C_{h_l}UT_{h_l}U^*D_{h_l}\| - \sum_{k\neq l}\|C_{h_l}UE_{h_k}\|\cdot\|D_{h_l}UE_{h_k}\| \\ &\geq \frac{\varepsilon^2}{16} - \frac{\varepsilon^2}{32} = \frac{\varepsilon^2}{32}.\end{aligned}$$

So, $\underline{\lim}_l\|C_{h_l}(UTU^* - \delta_{\mathcal{E}}(UTU^*))D_{h_l}\| = \underline{\lim}_l\|C_{h_l}UTU^*D_{h_l}\| > \varepsilon^2/32$ implies that $UTU^* - \delta_{\mathcal{E}}(UTU^*)$ is not compact by Proposition 3 and therefore that UTU^* does not belong to $\mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$ by Proposition 4. This concludes the argument. □

LEMMA 6. *Assume that P and Q are projections. Then*

$$\|P - Q\| = \max\{\|P^\perp Q\|, \|PQ^\perp\|\}.$$

PROOF. Since $P - Q = PQ^\perp - P^\perp Q$, it follows that $\|P - Q\| = \|PQ^\perp - P^\perp Q\|$. Because $PQ^\perp$ and $P^\perp Q$ have orthogonal initial and final spaces,

$$\|PQ^\perp - P^\perp Q\| = \max\{\|PQ^\perp\|, \|P^\perp Q\|\}. \quad \square$$

LEMMA 7. *If U is a selfadjoint unitary operator which implements an automorphism of $\mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$ then there exists a bijection σ of* $\mathbf{N}$ *such that $\|UE_iU^* - E_{\sigma(i)}\| \to 0$.*

PROOF. From Lemma 5 there exists a subset M_0 of $\mathbf{N}$ such that $\mathbf{N} - M_0$ is finite and for all $i \in M_0$ there exists a $\tau(i) \in \mathbf{N}$ for which $\|E_{\tau(i)}^\perp UE_i\|^2 < \frac{1}{2}$ and because this implies that $E_{\tau(i)}U|_{E_i\mathcal{H}}$ has no null space it follows that $\dim E_{\tau(i)} \geq \dim E_i$; that $\dim E_{\tau(i)} \leq \dim E_i$ for all but finitely many $i \in M_0$ follows from an argument similar to that in Lemma 5 which we include for completeness:

Contrapositively, we assume that $\dim E_{\tau(i)} \gneq \dim E_i$ for all i in an infinite subset S of M_0. Let i_1 be the first integer in S. Since U is unitary there is an integer n_1 such that

$$\left\|\left(E_{\tau(i_1)} \ominus \operatorname{ran} UE_{i_1}\right)U\left(\sum_{l=1, l\neq i_1}^{n_1} E_l\right)\right\| > \frac{1}{4}$$

whereas, since $i_1 \in M_0$, $\|E_{\tau(i_1)}UE_{i_1}\| \geq \frac{1}{4}$. Because $U = U^*$, $\|(\sum_{l=1;l\neq i_1}^{n_1} E_l)UE_{\tau(i_1)}\|$, $\|E_{i_1}UE_{\tau(i_1)}\| \geq \frac{1}{4}$. Assuming that you have selected $\{i_l\}_{l=1}^{k-1}$, let i_k be the next integer in S such that $i_k > n_{k-1}$ and $\|E_{\tau(i_k)}U(\sum_{l=1}^{n_{k-1}} E_l + \sum_{l=1}^{k-1} E_{i_l})\| < \frac{1}{4}$. There exists an integer n_k such that

$$\left\|(E_{\tau(i_k)} \ominus \operatorname{ran} UE_{i_k})U\left(\sum_{l=n_{k-1}+1,l\neq i_k}^{n_k} E_l\right)\right\| \geq \frac{1}{4}.$$

Continue inductively to obtain an infinite set of finite dimensional, mutually orthogonal projections $\{C_{i_k}, D_{i_k}\}$ where $C_{i_k} = \sum_{l=n_{k-1}+1}^{n_k} E_l$ and $D_{i_k} = E_{i_k}$ so that $\|C_{i_k}UE_{\tau(i_k)}\|$, $\|D_{i_k}UE_{\tau(i_k)}\| \geq \frac{1}{4}$ and apply the argument of Lemma 5 to arrive at a contradiction–thereby forcing $\dim E_{\tau(i)} = \dim E_i$ for all but finitely many $i \in M_0$. If we let $M = \{i \in M_0 : \dim E_i = \dim E_{\tau(i)}\}$ then we conclude that $\mathbf{N} - M$ is finite.

We make the following simple observation: if E and F are projections and U is unitary, then for $x \in E\mathcal{H}$ with $\|x\| = 1$, $\|FUEx\|^2 + \|F^\perp UEx\|^2 = 1$ so that

$$1 - \inf_{\substack{x\in E\mathcal{H}\\ \|x\|=1}} \|FUEx\|^2 = \sup_{\substack{x\in E\mathcal{H}\\ \|x\|=1}} \|F^\perp UEx\|^2 = \|F^\perp UE\|^2 \quad (*).$$

Next we observe that since $E_{\tau(i)}U|_{E_i\mathcal{H}} \in \mathcal{L}(E_i\mathcal{H}, E_{\tau(i)}\mathcal{H})$ is invertible, $E_{\tau(i)}U|_{E_i\mathcal{H}} = VP$ where $P \in \mathcal{L}(E_i\mathcal{H})$ is positive and $V \in \mathcal{L}(E_i\mathcal{H}, E_{\tau(i)}\mathcal{H})$ is unitary by the polar decomposition; since $U = U^*$, $E_iU|_{E_{\tau(i)}\mathcal{H}} = PV^*$. We use this fact to show that $\|E_{\tau(i)}UE_i^\perp\| = \|E_{\tau(i)}^\perp UE_i\|$ for all $i \in M$:

$$\|E_{\tau(i)}UE_i^\perp\|^2 = \|E_i^\perp UE_{\tau(i)}\|^2 = \sup_{\substack{x\in E_{\tau(i)}\mathcal{H}\\ \|x\|=1}} \|E_i^\perp UE_{\tau(i)}x\|^2 \qquad \text{because } U = U^*$$

$$= 1 - \inf_{\substack{x\in E_{\tau(i)}\mathcal{H}\\ \|x\|=1}} \|E_iUE_{\tau(i)}x\|^2 \qquad \text{by } (*)$$

$$= 1 - \inf_{\substack{x\in E_{\tau(i)}\mathcal{H}\\ \|x\|=1}} \|PV^*x\|^2 = 1 - \inf_{\substack{y\in E_i\mathcal{H}\\ \|y\|=1}} \|Py\|^2 \qquad \text{because } V^* \text{ is unitary}$$

$$= 1 - \inf_{\substack{y\in E_i\mathcal{H}\\ \|y\|=1}} \|VPy\|^2 = 1 - \inf_{\substack{y\in E_i\mathcal{H}\\ \|y\|=1}} \|E_{\tau(i)}UE_iy\|^2 = \|E_{\tau(i)}^\perp UE_i\|^2 \qquad \text{by } (*).$$

From the construction of M and Lemma 6 we conclude that

$$\|UE_iU^* - E_{\tau(i)}\| = \max\{\|E_{\tau(i)}^\perp UE_i\|, \|E_{\tau(i)}UE_i^\perp\|\} \to 0.$$

We let $\sigma(i) = \tau(i)$ for all $i \in M$ and let $\sigma(i) = i$ for all $i \notin M$ so that to obtain the result it remains only to show that τ is a bijection of M. But, for each $i \in M$,

$$\tfrac{1}{2} > \|UE_iU - E_{\tau(i)}\| = \|U^2E_iU^2 - UE_{\tau(i)}U\| = \|E_i - UE_{\tau(i)}U\| = \max\{\|E_i^{\perp}UE_{\tau(i)}\|, \|E_iUE_{\tau(i)}^{\perp}\|\}$$

so that $\dim E_i = \dim UE_{\tau(i)}U = \dim E_{\tau(i)}$ and $\|E_i^{\perp}UE_{\tau(i)}\| < \frac{1}{2}$. We remark that $\tau^2(i) = i$ for all i in M because the value of τ is uniquely determined since $\|U\| = 1$ and $\inf_{x \in E_i\mathcal{H}; \|x\|=1}\|E_{\tau(i)}UE_ix\|^2 > \frac{1}{2}$ by (*). Hence, τ is a one-to-one map of M onto M, thus concluding our argument. □

CONSTRUCTION 8. Suppose U is a selfadjoint unitary operator which implements an automorphism of $\mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$. From U we will construct a selfadjoint unitary V which implements an automorphism of $\mathfrak{D}(\mathcal{E})$ such that $\|UE_jU^* - VE_jV^*\| \to 0$.

PROOF. Let M be the set and σ the bijection of $\mathbf{N}$ introduced in Lemma 7. To define $V \sim (V_{ij})$, let $V_{ij} \equiv 0$ whenever $i \neq \sigma(j)$ and let $V_{\sigma(j)j} = 1$ whenever $j \notin M$. For $j \in M$ recall that $\dim E_j = \dim E_{\sigma(j)}$ so that $E_{\sigma(j)}U|_{E_j\mathcal{H}}$ can be identified as the matrix of an operator from one n-dimensional Hilbert space to another with polar decomposition $W_{\sigma(j)j}P_{\sigma(j)j}$; since $E_{\sigma(j)}U|_{E_j\mathcal{H}}$ is invertible for $j \in M$, $W_{\sigma(j)j}$ is unitary. We proceed, inductively, as follows: Suppose j_1 is the first integer in M. Let $V_{\sigma(j_1)j_1} = W_{\sigma(j_1)j_1}$ and let $V_{j_1,\sigma(j_1)} = (W_{\sigma(j_1)j_1})^*$. Let j_2 be the next integer in M such that $j_2 \notin \{j_1, \sigma(j_1)\}$ and continue as before. Clearly V is unitary and implements an automorphism of $\mathfrak{D}(\mathcal{E})$. To verify that V is selfadjoint, it suffices to show that for those $j \in M$ such that $\sigma(j) = j$, $V_{jj} = (V_{jj})^*$. Since U is selfadjoint so is U_{jj}, so that $W_{jj}P_{jj} = P_{jj}(W_{jj})^*$ by definition and $W_{jj}P_{jj} = P_{jj}W_{jj}$ by [6, p. 69]. Since P_{jj} is invertible (because U_{jj} is), we conclude that $W_{jj} = (W_{jj})^*$ and $V_{jj} = (V_{jj})^*$ by construction. Finally, since $VE_jV^* = E_{\sigma(j)}$ for each j, we have that $\|UE_jU^* - VE_jV^*\| = \|UE_jU^* - E_{\sigma(j)}\| \to 0$ following Lemma 7. □

In the lemmas which follow, we adopt common usage and say that an operator $T \in \mathcal{L}(\mathcal{H})$ belongs to the *essential commutant* of a C^*-subalgebra $\mathfrak{A}$ of $\mathcal{L}(\mathcal{H})$ if and only if $TS - ST \in \mathcal{C}(\mathcal{H})$ for every $S \in \mathfrak{A}$.

LEMMA 9. *Suppose that W is a unitary operator which implements an automorphism of $\mathfrak{A} = \mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$ such that $\|WE_j - E_jW\| \to 0$. Then W belongs to the essential commutant of the center of $\mathfrak{D}(\mathcal{E})$.*

PROOF. It suffices to assume that T belongs to the center of $\mathfrak{D}(\mathcal{E})$ and then show that $T - WTW^*$ is compact. Since W implements an automorphism of $\mathfrak{A}$, WTW^* belongs to $\mathfrak{A}$. Hence, $\delta_{\mathcal{E}}[WTW^*] - WTW^*$ is compact by Proposition 4, and to prove the assertion, it suffices to show that $T - \delta_{\mathcal{E}}[WTW^*]$ is compact; since both terms belong to $\mathfrak{D}(\mathcal{E})$, this is equivalent to showing that $\|E_j[T - WTW^*]E_j\| \to 0$. Since T belongs to the center of $\mathfrak{D}(\mathcal{E})$, $T = \sum_{j=1}^{\infty}\lambda_jE_j$ for a bounded sequence of complex numbers $\{\lambda_j\}_{j=1}^{\infty}$; thus,

$$\begin{aligned}\|E_j[T - WTW^*]E_j\| &= \|\lambda_j E_j - E_j WTW^* E_j\| \\ &\leq \|\lambda_j E_j - E_j WTE_j W^*\| + \|E_j WT(W^* E_j - E_j W^*)\| \\ &\leq \|\lambda_j E_j - W(\lambda_j E_j)W^*\| + \|T\| \cdot \|W^* E_j - E_j W^*\| \\ &\leq |\lambda_j| \cdot \|W^* E_j - E_j W^*\| + \|T\| \cdot \|W^* E_j - E_j W^*\| \to 0.\end{aligned}$$

Hence, W essentially commutes with every operator in the center of $\mathfrak{D}(\mathcal{E})$. □

LEMMA 2′. *Let U be a selfadjoint unitary operator which implements an automorphism of $\mathcal{Q} = \mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$. Then there is a selfadjoint unitary operator V which implements an automorphism of $\mathfrak{D}(\mathcal{E})$ such that U^*V is a compact perturbation of the identity.*

PROOF. If U is any selfadjoint unitary operator which implements an automorphism of $\mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$, then by Construction 8 we obtain an operator V such that U^*V implements an automorphism of $\mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$ and $\|U^*VE_j - E_jU^*V\| \to 0$. Hence by Lemma 9, U^*V belongs to the essential commutant of the center of $\mathfrak{D}(\mathcal{E})$; by an application of [8, Theorem 2.1] it follows that U^*V belongs to [center $(\mathfrak{D}(\mathcal{E}))]' + \mathcal{C}(\mathcal{H})$. Consequently, U^*V belongs to $\mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$. Now

$$\begin{aligned}1 - U^*V &= \delta_{\mathcal{E}}(1 - U^*V) + [1 - U^*V - \delta_{\mathcal{E}}(1 - U^*V)] \\ &= \delta_{\mathcal{E}}(1 - U^*V) - [U^*V - \delta_{\mathcal{E}}(U^*V)],\end{aligned}$$

where $U^*V - \delta_{\mathcal{E}}(U^*V)$ is compact by Proposition 4. So, to prove the assertion, it suffices to show that $\delta_{\mathcal{E}}(1 - U^*V)$ is compact. But, that follows from a matrix multiplication: if $j \in M$, then

$$\begin{aligned}\|E_j(1 - U^*V)E_j\| &= \|E_j - E_j(U^*)_{j,\sigma(j)}V_{\sigma(j),j}E_j\| \\ &= \|E_j - E_j(U_{\sigma(j),j})^*V_{\sigma(j),j}E_j\| \\ &= \|E_j - E_j(P_{\sigma(j),j})(W_{\sigma(j),j})^*W_{\sigma(j),j}E_j\| \\ &= \|E_j - E_j(P_{\sigma(j),j})E_j\| \\ &= \|E_j[1_{E_j\mathcal{H}} - (P_{\sigma(j),j})]E_j\| \\ &\leq \|1_{E_j\mathcal{H}} - P_{\sigma(j),j}\|.\end{aligned}$$

We remark that $P_{\sigma(j),j}$ is a positive invertible operator on the finite dimensional space $E_j\mathcal{H}$. So for $j \in M$

$$\begin{aligned}\|1_{E_j\mathcal{H}} - P_{\sigma(j),j}\| &= 1 - \inf\{\mu : \mu \in \text{spectrum}(P_{\sigma(j),j})\}\\ &= 1 - \inf_{\substack{x\in E_j\mathcal{H}\\ \|x\|=1}} \|P_{\sigma(j),j}x\|\\ &= 1 - \inf_{\substack{x\in E_j\mathcal{H}\\ \|x\|=1}} \|W_{\sigma(j),j}P_{\sigma(j),j}x\| \qquad \text{since } W_{\sigma(j),j} \text{ is unitary}\\ &\leqslant 1 - \inf_{\substack{x\in E_j\mathcal{H}\\ \|x\|=1}} \|E_{\sigma(j)}UE_jx\|^2 \to 0\end{aligned}$$

$$\text{since } \inf_{\substack{x\in E_j\mathcal{H}\\ \|x\|=1}} \|E_{\sigma(j)}UE_jx\|^2 = 1 - \|E^{\perp}_{\sigma(j)}UE_j\|^2.$$

Because $\mathbf{N} - M$ is finite, it follows that $\|E_j(1 - U^*V)E_j\| \to 0$, or equivalently, that $\delta_{\mathcal{E}}(1 - U^*V)$ is compact, from which we conclude that $1 - U^*V$ is compact. □

REMARK 10. Suppose that W is a unitary operator which implements an automorphism of $\mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$. Then W is a compact perturbation of a partial isometry Y of index zero such that $\{YE_iY^*\}_{i=1}^{\infty} \subset \mathcal{E} \cup \{0\}$.

PROOF. Adopting the notation of Theorem 1, we let $\mathfrak{D}(\mathcal{F}) = \mathfrak{D}(\mathcal{E})$ and P be the projection onto the first coordinate space of $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$. So, $W = P^{\perp}U|_{P\mathcal{K}} = P^{\perp}V|_{P\mathcal{K}} - P^{\perp}Q|_{P\mathcal{K}}$; let $Y = P^{\perp}V|_{P\mathcal{K}}$ and the assertion is easily verified after noting that $\ker(Y) = (\Sigma_{i\in N_0}E_i)\mathcal{H}$ and $\ker(Y^*) = (\Sigma_{j\in N_1}E_j)\mathcal{H}$. □

DEFINITION. Let W be a unitary operator which implements an automorphism of $\mathcal{Q} + \mathcal{C}(\mathcal{H})$, where $\mathcal{Q}$ is a C^*-algebra. We say that W *splits* (with respect to $\mathcal{Q}$) if it is a compact perturbation of a unitary operator which implements an automorphism of $\mathcal{Q}$.

It should be mentioned that Lemma 2 is not a surprising result as it is suggested by [8, Theorem 4.10] which asserts that *every* unitary which implements an automorphism of $\mathcal{Q} + \mathcal{C}(\mathcal{H})$ splits whenever $\mathcal{Q}$ is a maximal abelian selfadjoint subalgebra of $\mathcal{L}(\mathcal{H})$. However, our situation differs in that there are unitary operators which implement automorphisms of perturbed block diagonal algebras which do not split.

In the following corollaries, P_k will denote $\Sigma\{E_i: \dim E_i = k\}$.

COROLLARY 11. *Let U be a unitary operator which implements an automorphism of $\mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$. U splits if and only if*

$$\text{index}(P_kU|_{P_k\mathcal{H}}) = 0 \quad \textit{for every } k \in \mathbf{N}.$$

PROOF. ⇒: By assumption there exists a unitary operator V which commutes with P_k for each k and a compact C such that $U = V + C$. Since index is invariant under compact perturbation [4],

$$\operatorname{index}(P_k U|_{P_k\mathcal{H}}) = \operatorname{index}(P_k V|_{P_k\mathcal{H}}) = 0.$$

$\Leftarrow$: From Remark 10, $U = W + C$, where C is compact and W is a partial isometry of index zero such that $\{WE_iW^*\}_{i=1}^{\infty} \subset \mathcal{E} \cup \{0\}$. Note that $P_k W|_{P_k\mathcal{H}}$ is unitary for all $k \notin R$, where R is a finite subset of N; for each $k \in R$, $\operatorname{index}(P_k W|_{P_k\mathcal{H}}) = 0$, so

$$\ker(P_k W|_{P_k\mathcal{H}}) = \sum_{i \in N_k} E_i \quad \text{and}$$

$$\operatorname{cokernel}(P_k W|_{P_k\mathcal{H}}) = \sum_{i \in 0_k} E_i \qquad \text{with } \operatorname{card}(N_k) = \operatorname{card}(0_k) < \infty.$$

Define F_k, a finite rank partial isometry such that $\{F_k E_i F_k^*\}_{i \in N_k} = \{E_i\}_{i \in 0_k}$, and note that U is a compact perturbation of a unitary operator $W + \Sigma_{k \in R} F_k$ which implements an automorphism of $\mathfrak{D}(\mathcal{E})$. □

In Corollary 11 we presented a necessary and sufficient condition for a specific unitary operator to split. In Corollary 12 we obtain a necessary and sufficient condition for a perturbed block diagonal algebra to have the property that *every* unitary implementing an automorphism of it splits:

COROLLARY 12. *Every unitary operator which implements an automorphism of $\mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$ splits if and only if at most one projection P_k is infinite dimensional.*

PROOF. $\Rightarrow$: We assume that P_k and P_l are infinite dimensional for $k \neq l$ and construct a unitary U that implements an automorphism of $\mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$ but does not split. For each j, let $\operatorname{support}(P_j) = \{i\colon \dim E_i = j\}$ and let $\{M_i\}_{i=1}^{\infty}$ and $\{N_i\}_{i=1}^{\infty}$ be partitions of $\operatorname{support}(P_k)$ and $\operatorname{support}(P_l)$ respectively such that $\operatorname{card} M_i = l$ and $\operatorname{card} N_i = k$ for all i. Let U be any unitary operator such that $UE_iU^* = E_i$ for $i \notin \operatorname{support}(P_k) \cup \operatorname{support}(P_l)$ and

$$U\left(\sum_{i \in M_1} E_i\right)U^* = \sum_{i \in N_1} E_i,$$

$$\{UE_iU^*\}_{i \in M_k} = \{E_i\}_{i \in M_{k-1}}, \qquad k > 1,$$

$$\{UE_iU^*\}_{i \in N_k} = \{E_i\}_{i \in N_{k+1}}, \qquad k \geqslant 1,$$

and note that U implements an automorphism of $\mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$, but that $\operatorname{index}(P_k U|_{P_k\mathcal{H}}) = kl \neq 0$, so that by Corollary 11, U does not split.

$\Leftarrow$: This is a routine argument modulo Remark 10. □

We recall that an automorphism of an algebra is said to be *inner* if it can be implemented by a unitary which belongs to the algebra.

COROLLARY 13. *All automorphisms of $\mathfrak{A} = \mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$ are inner if and only if there exists a positive integer l such that whenever $n,m > l$, $n \neq m$, then* $\dim(E_n) \neq \dim(E_m)$.

PROOF. $\Leftarrow$: Every automorphism α of $\mathcal{A}$ is implemented by a unitary operator U_α (by a slight variation of [8, Lemma 4.5]). By Remark 10, $U_\alpha = W + C$, where C is compact and W is a partial isometry of index zero such that $\{WE_iW^*\}_{i=1}^{\infty} \subseteq \mathcal{E} \cup \{0\}$ so it follows that U belongs to $\mathcal{A}$ iff W does. Since W must map the E_i to other E_i of the same dimension, the assumption implies that $WE_iW^* = E_i$ for all but finitely many $i > l$ and that W belongs to $\mathcal{A}$.

$\Rightarrow$: In this proof by the contrapositive, we assume that for every positive integer l there exist $n,m > l$, $n \neq m$, such that $\dim E_n = \dim E_m$ and then assert that there is a unitary operator U which implements an automorphism of $\mathcal{D}(\mathcal{E})$ (hence, $\mathcal{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$) but which does not belong to $\mathcal{A}$. We leave this construction to the reader and then remark that the automorphism α which is induced by U cannot be implemented by any unitary operator which belongs to $\mathcal{A}$ and therefore is not inner. $\square$

COROLLARY 14. *Suppose* $\mathcal{A} = \mathcal{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$ *and* $\mathcal{B} = \mathcal{D}(\mathcal{F}) + \mathcal{C}(\mathcal{H})$. *The following are equivalent*:

(i) $\mathcal{A} = \mathcal{B}$.

(ii) *There exist finite subsets* O_1 *and* O_2 *of* $\mathbf{N}$, *a unitary operator* U *which is a compact perturbation of the identity, and a bijection* $\sigma\colon \mathbf{N} - O_1 \to \mathbf{N} - O_2$ *such that* $U(\sum_{i\in O_1} E_i)U^* = \sum_{j \in O_2} F_j$ *and* $UE_iU^* = F_{\sigma(i)}$ *for* $i \in \mathbf{N} - O_1$.

(iii) *There exist finite subsets* O_1 *and* O_2 *of* $\mathbf{N}$ *and a bijection* $\sigma\colon \mathbf{N} - O_1 \to \mathbf{N} - O_2$ *which satisfies this condition*: *for every* $\varepsilon > 0$ *there is a finite subset* $N(\varepsilon)$ *of* $\mathbf{N}$ *such that*

$$\left\| \sum_{i \notin N(\varepsilon) \cup O_1} \lambda_i E_i - \sum_{i \notin N(\varepsilon)\cup O_1} \lambda_i F_{\sigma(i)} \right\| < \varepsilon$$

for every sequence $\{\lambda_i\}_{i \notin N(\varepsilon)\cup O_1}$ *of zeros and ones.*

PROOF. (i) $\Rightarrow$ (ii): $\mathcal{A} = \mathcal{B}$ implies that $\mathcal{A} \simeq \mathcal{B}$ and by Theorem 1 the existence of a unitary operator V such that for finite sets N_1 and N_2 and a bijection $\gamma\colon \mathbf{N} - N_1 \to \mathbf{N} - N_2$, $V(\sum_{i\in N_1} E_i)V^* = \sum_{j \in N_2} F_j$ and $VE_iV^* = F_{\gamma(i)}$ for all $i \notin N_1$. Since V implements an automorphism of $\mathcal{A}$, $V = W + Q$, where $Q \in \mathcal{C}(\mathcal{H})$ and W is a unitary operator for which there exist finite subsets M_1 and M_2 of $\mathbf{N}$ and a bijection $\tau\colon \mathbf{N} - M_1 \to \mathbf{N} - M_2$ such that $W(\sum_{i\in M_1} E_i)W^* = \sum_{i\in M_2} E_i$ and $WE_iW^* = E_{\tau(i)}$ for all $i \notin M_1$. Hence $VW^* = 1 + QW^*$ and the reader may verify that if $O_1 = M_2 \cup \tau(N_1 - N_1 \cap M_1)$ and $O_2 = N_2 \cup \gamma(M_1 - M_1 \cap N_1)$ then $\sigma = \gamma \circ \tau^{-1}$ is a bijection from $\mathbf{N} - O_1$ to $\mathbf{N} - O_2$ such that if $U = VW^*$, $U(\sum_{i \in O_1} E_i)U^* = \sum_{j\in O_2} F_j$ and $UE_iU^* = F_{\sigma(i)}$ for all $i \notin O_1$.

(ii) $\Rightarrow$ (iii): Since $U = 1 + K$ for K compact,

$$\sum_{i\notin O_1} \lambda_i F_{\sigma(i)} = U\left(\sum_{i\notin O_1} \lambda_i E_i\right)U^* = \sum_{i\notin O_1} \lambda_i E_i + K\left(\sum_{i\notin O_1} \lambda_i E_i\right)$$
$$+ \left(\sum_{i\notin O_1} \lambda_i E_i\right)K^* + K\left(\sum_{i\notin O_1} \lambda_i E_i\right)K^*$$

and

$$\left\|\sum_{i\notin O_1} \lambda_i F_{\sigma(i)} - \sum_{i\notin O_1} \lambda_i E_i\right\| \leq 3\max\{1, \|K\|\} \cdot \left\|K\left(\sum_{i\notin O_1} \lambda_i E_i\right)\right\|$$

for every sequence $\{\lambda_i\}$ of zeros and ones. Because K is compact, given $\varepsilon > 0$, there exists a subset $N(\varepsilon)$ such that

$$\left\|K\left(\sum_{i\notin N(\varepsilon)\cup O_1} \lambda_i E_i\right)\right\| < \frac{\varepsilon}{3\max\{1, \|K\|\}}$$

for every sequence of zeros and ones.

(iii) $\Rightarrow$ (i): We assert that $\sum_{i\notin O_1}\lambda_i E_i - \sum_{i\notin O_1}\lambda_i F_{\sigma(i)}$ is compact for every sequence $\{\lambda_i\}_{i\notin O_1}$ of zeros and ones; for, let

$$P_k = \left(\sum_{i\in O_1} E_i\right) \vee \left(\sum_{j\in O_2} F_j\right) \vee E_{i_1} \vee F_{\sigma(i_1)} \cdots \vee E_{i_k} \vee F_{\sigma(i_k)},$$

where i_k is the kth integer in $\mathbf{N} - O_1$, and note that $P_k^\perp \searrow 0$, and that

$$\left\|\left(\sum_{i\notin O_1} \lambda_i E_i - \sum_{i\notin O_1} \lambda_i F_{\sigma(i)}\right)P_k^\perp\right\| \leq \left\|\sum_{\substack{i\notin O_1\\ i>k}} \lambda_i E_i - \sum_{\substack{i\notin O_1\\ i>k}} \lambda_i F_{\sigma(i)}\right\| \to 0$$

by (iii). It is not difficult to show that the essential center of $\mathcal{A}$ is generated by $\{\sum_{i=1}^\infty \lambda_i E_i\colon \lambda_i = 0 \text{ or } 1\} + \mathcal{C}(\mathcal{H})$ and to conclude that the essential center of $\mathcal{A}$ is equal to the essential center of $\mathcal{B}$ so that by [8, Theorem 2.1], $\mathcal{A} = \mathcal{B}$. □

Since $\mathcal{C}(\mathcal{H})$ is a norm-closed, two sided *-ideal in $\mathcal{L}(\mathcal{H})$, the quotient $\mathcal{L}(\mathcal{H})/\mathcal{C}(\mathcal{H})$ is a C^*-algebra commonly referred to as the Calkin algebra and $\pi\colon \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathcal{C}(\mathcal{H})$ is the usual projection map, taking T to its coset $\{T + K\colon K \in \mathcal{C}(\mathcal{H})\}$.

Definition. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are C^*-subalgebras of the Calkin algebra. $\mathcal{A}$ and $\mathcal{B}$ are said to be *essentially unitarily equivalent* if and only if some unitary operator in the Calkin algebra implements an isomorphism of them.

Corollary 15. *Let* $\mathcal{A} = \mathcal{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$ *and* $\mathcal{B} = \mathcal{D}(\mathcal{F}) + \mathcal{C}(\mathcal{H})$. $\pi(\mathcal{A})$ *and* $\pi(\mathcal{B})$ *are essentially unitarily equivalent if and only if there exist finite sets*

N_0 and N_1 and a bijection τ: $\mathbf{N} - N_0 \to \mathbf{N} - N_1$ such that $\dim(E_i) = \dim(F_{\tau(i)})$ *for every $i \in \mathbf{N} - N_0$.*

PROOF. $\Leftarrow$: Let W be a partial isometry with $\ker W = \sum_{i \in N_0} E_i$ and $\operatorname{coker} W = \sum_{j \in N_1} F_j$ such that $WE_iW^* = F_{\tau(i)}$ for all $i \in \mathbf{N} - N_0$. Clearly, $\pi(W)$ is unitary and $\pi(W)\pi(\mathcal{A})\pi(W)^* = \pi(\mathcal{B})$.

$\Rightarrow$: Assuming that $\pi(W)$ implements an isomorphism of $\pi(\mathcal{A})$ and $\pi(\mathcal{B})$, then $U = \begin{pmatrix} 0 & W^* \\ W & 0 \end{pmatrix}$ is selfadjoint and $\pi(U)$ is unitary; one can verify that U is a compact perturbation of a selfadjoint unitary R and that this selfadjoint unitary implements an automorphism of $\mathfrak{D}(\mathcal{E}) \oplus \mathfrak{D}(\mathcal{F}) + \mathcal{C}(\mathcal{H} \oplus \mathcal{H})$. From Lemma 2, R is a compact perturbation of a selfadjoint unitary V which implements an automorphism of $\mathfrak{D}(\mathcal{E}) \oplus \mathfrak{D}(\mathcal{F})$. Now adopting the notation and arguing as in the proof of Theorem 1, since τ is dimension-preserving it suffices to observe that N_0 and N_1 are finite subsets. □

We now shift our attention to a sequence $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ of finite dimensional projections increasing to the identity and to the algebras which are naturally associated with it. The *triangular algebra associated with* $\mathcal{P}$ and denoted as $\mathfrak{T}(\mathcal{P})$ is defined to be the set of those operators which leave all of the projections of $\mathcal{P}$ invariant. The *quasitriangular algebra associated with* $\mathcal{P}$ and denoted as $\mathfrak{Q}\mathfrak{T}(\mathcal{P})$ is defined to be the set of operators T for which $\|P_n^{\perp}TP_n\| \to 0$. It is easy to verify that $\mathfrak{T}(\mathcal{P}) + \mathcal{C}(\mathcal{H})$ is contained in $\mathfrak{Q}\mathfrak{T}(\mathcal{P})$; W. Arveson proved, in fact, that $\mathfrak{T}(\mathcal{P}) + \mathcal{C}(\mathcal{H}) = \mathfrak{Q}\mathfrak{T}(\mathcal{P})$ [1].

DEFINITION. An algebra $\mathcal{A}$ is called *quasidiagonal for a sequence* $\mathcal{P}$, and denoted as $\mathfrak{Q}\mathfrak{D}(\mathcal{P})$, if it consists of those operators T for which $\|TP_n - P_nT\| \to 0$.

If $\mathfrak{D}(\mathcal{P})$ denotes the set of those operators which commute with the elements of $\mathcal{P}$, then it is easy to see that $\mathfrak{D}(\mathcal{P}) + \mathcal{C}(\mathcal{H})$ is simply a perturbed block diagonal algebra and is contained in $\mathfrak{Q}\mathfrak{D}(\mathcal{P})$. It was already known that for every T in $\mathfrak{Q}\mathfrak{D}(\mathcal{P})$ there is a subsequence $\{P_{n_k}\}_{k=1}^{\infty}$ of $\mathcal{P}$, dependent on T, for which T belongs to $\mathfrak{D}(\{P_{n_k}\}_{k=1}^{\infty}) + \mathcal{C}(\mathcal{H})$ [7]. The natural question, suggested by the nonselfadjoint case, is whether $\mathfrak{D}(\mathcal{P}) + \mathcal{C}(\mathcal{H})$ equals $\mathfrak{Q}\mathfrak{D}(\mathcal{P})$. In Lemma 17 we show that it does not.

PROPOSITION 16. $\mathfrak{Q}\mathfrak{D}(\mathcal{P}) = \mathfrak{Q}\mathfrak{T}(\mathcal{P}) \cap \mathfrak{Q}\mathfrak{T}(\mathcal{P})^*$.

PROOF. $\subseteq$: If $T \in \mathfrak{Q}\mathfrak{D}(\mathcal{P})$ then $\|TP_n - P_nT\| \to 0$ so $\|P_n^{\perp}TP_n\| = \|P_n^{\perp}(TP_n - P_nT)\| \to 0$; by applying the same argument to the adjoint of $TP_n - P_nT$, we have that $\|P_n^{\perp}T^*P_n\| = \|P_nTP_n^{\perp}\| \to 0$. Hence, $T \in \mathfrak{Q}\mathfrak{T}(\mathcal{P}) \cap \mathfrak{Q}\mathfrak{T}(\mathcal{P})^*$.

$\supseteq$: If $T \in \mathfrak{Q}\mathfrak{T}(\mathcal{P}) \cap \mathfrak{Q}\mathfrak{T}(\mathcal{P})^*$ then $\|TP_n - P_nT\| \leqslant \|(1 - P_n)TP_n\| + \|P_nT(1 - P_n)\| \to 0$, so $T \in \mathfrak{Q}\mathfrak{D}(\mathcal{P})$. □

DEFINITION. Let $\mathcal{H}$ be a Hilbert space and M be the matrix of an operator with respect to a fixed orthonormal basis $\{e_i\}$. Define $+_{\mathcal{H}}$: $M \to M^+$ to be the linear map on matrices which takes M to a matrix derived from it by replacing all entries below the main diagonal with zeros. Such maps have been studied before [5], [10].

LEMMA 17. $\mathfrak{D}(\mathcal{P}) + \mathcal{C}(\mathcal{H}) \subsetneq \mathcal{Q}\mathfrak{D}(\mathcal{P})$.

PROOF. To simplify the notation, we assume first that $\dim P_n = n$ for all n. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal set in $\mathcal{H}$ such that $e_n \in (P_n \ominus P_{n-1})\mathcal{H}$ for each n. Let $\mathcal{H}_i = \text{span}\{e_k: i^2 \leq k < (i+1)^2\}$ and for each i let $C_i \in \mathcal{L}(\mathcal{H}_i)$ with $\|C_i^+\| = 1$ and $\|C_i\| = \| +_i\|^{-1}$. Put $C = C_1 \oplus C_2 \oplus \cdots$ so that $C^+ = C_1^+ \oplus C_2^+ \oplus \cdots$ is not compact; however, $\dim \mathcal{H}_i \to \infty$ so that $\| +_i \| \to \infty$ [9, Proposition 1.2] and C is compact. Hence $\delta(C^+) = \delta(C)$ is compact so $C^+ - \delta(C^+)$ is not compact and $C^+ \not\in \mathfrak{D}(\mathcal{P}) + \mathcal{C}(\mathcal{H})$. However, $C^+ \in \mathfrak{T}(\mathcal{P})$ and $C^+ = C - (C - C^+) \in \mathfrak{T}(\mathcal{P})^* + \mathcal{C}(\mathcal{H})$ because $C - C^+ \in \mathfrak{T}(\mathcal{P})^*$; hence, $C^+ \in \mathcal{Q}\mathfrak{D}(\mathcal{P})$ by Proposition 16. To generalize the argument to an arbitrary sequence $\mathcal{P}$, construct C_i as before and define $D_i \in \mathcal{L}(P_{(i+1)^2}\mathcal{H} \ominus P_{i^2}\mathcal{H})$ by $D_i = C_i \oplus 0$ so that $D = D_1 \oplus D_2 \oplus \cdots$ is compact and D^+ belongs to $\mathcal{Q}\mathfrak{D}(\mathcal{P})$ but not to $\mathfrak{D}(\mathcal{P}) + \mathcal{C}(\mathcal{H})$. □

The interested reader can verify that $\mathfrak{D}(\mathcal{P}) + \mathcal{C}(\mathcal{H})$ is equal to the norm closure of $[\mathfrak{T}(\mathcal{P}) + \mathcal{I}] \cap [\mathfrak{T}(\mathcal{P})^* + \mathcal{I}]$, where $\mathcal{I}$ is the ideal of Hilbert-Schmidt operators. We remark that Lemma 17 together with Proposition 16 and the fact that $\mathcal{Q}\mathfrak{T}(\mathcal{P}) = \mathfrak{T}(\mathcal{P}) + \mathcal{C}(\mathcal{H})$ [1] says that $\mathfrak{D}(\mathcal{P}) + \mathcal{C}(\mathcal{H}) \subsetneq [\mathfrak{T}(\mathcal{P}) + \mathcal{C}(\mathcal{H})] \cap [\mathfrak{T}(P)^* + \mathcal{C}(\mathcal{H})]$. Nevertheless, we can obtain $\mathfrak{D}(\mathcal{P}) + \mathcal{C}(\mathcal{H})$ as the intersection of certain quasidiagonal algebras associated with the sequence $\mathcal{P}$. To do so, let π be any permutation of the positive integers and let $E_n = P_n \ominus P_{n-1}$ for each positive n; then P_n^π will denote $\Sigma\{E_i: \pi(i) \leq n\}$ and $\mathcal{Q}\mathfrak{D}^\pi(\mathcal{P})$ will denote the quasidiagonal algebra of those operators T for which $\|P_n^\pi T - TP_n^\pi\| \to 0$.

THEOREM 18. $\mathfrak{D}(\mathcal{P}) + \mathcal{C}(\mathcal{H}) = \bigcap_\pi \mathcal{Q}\mathfrak{D}^\pi(\mathcal{P})$, *where we intersect over all permutations* π.

PROOF. Clearly $\subseteq$ holds. To verify $\supseteq$, it suffices to assume $T \not\in \mathfrak{D}(\mathcal{P}) + \mathcal{C}(\mathcal{H})$ and then show that $T \not\in \bigcap_\pi \mathcal{Q}\mathfrak{D}^\pi(\mathcal{P})$. By [8, Theorem 2.1] we know that there exists an operator $Q \in \mathfrak{D}(\mathcal{P})'$ such that $QT - TQ \not\in \mathcal{C}(\mathcal{H})$; since every operator in $\mathfrak{D}(\mathcal{P})'$ can be approximated in norm by finite linear combinations of projections in $\mathfrak{D}(\mathcal{P})'$, we can assume that Q is a projection. Hence, either $QTQ^\perp$ or $Q^\perp TQ$ fails to be compact and without loss of generality we assume the former. So, there exist $\varepsilon > 0$ and a sequence of finite dimensional, mutually orthogonal projections $\{F_i, G_i\}_{i=1}^{\infty} \subset \mathfrak{D}(\mathcal{P})'$ such that $F_i = \Sigma_{k \in M_i} E_k < Q$ and $G_i = \Sigma_{k \in N_i} E_k < Q^\perp$ for all i and $\inf_i \|F_i T G_i\| \geq \varepsilon$. Now we define a permutation π of $\mathbf{N}$ for which $T \notin \mathcal{Q}\mathfrak{D}^\pi(\mathcal{P})$. Let π map N_1 to the first card(N_1) integers and map M_1 to the next card(M_1) integers. Let $R_1 = \{j \in \mathbf{N}: j \notin \cup_{i \in \mathbf{N}}(M_i \cup N_i)$ and $j < \text{card}(N_1) + \text{card}(M_1)\}$. Map R_1 to the next card(R_1) integers. Continue inductively with

$$R_{k+1} = \left\{ J \in \mathbf{N}: j \notin \left(\bigcup_{i \in \mathbf{N}} (M_i \cup N_i) \right) \cup_{i < k} R_i \right.$$

$$\left. \text{and } j < \sum_{i < k+1} (\text{card}(M_i) + \text{card}(N_i)) \right\}$$

and note that π is a bijection of N and that $T \notin \mathfrak{Q}\mathfrak{D}^{\pi}(\mathcal{P})$. □

If $\mathcal{S}$ is a subset of $\mathcal{L}(\mathcal{H})$, then we define lat$_f(\mathcal{S})$ to be the set of finite dimensional projections P on $\mathcal{H}$ such that $P^{\perp}TP = 0$ for every T in $\mathcal{S}$. The reader can easily verify that if $\mathcal{E} = \{E_n\}_{n=1}^{\infty}$ is a sequence of mutually orthogonal, finite dimensional projections on $\mathcal{H}$ whose sum is the identity, that R belongs to lat$_f \mathfrak{D}(\mathcal{E})$ if and only if R is a finite sum of E_n's. We view lat$_f \mathfrak{D}(\mathcal{E})$, ordered by range inclusion, as a directed set and $R \to \|R^{\perp}TR\|$ as a net on that directed set. We offer this intrinsic characterization of a perturbed block diagonal algebra based on Theorem 18:

COROLLARY 19.

$$T \in \mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H}) \quad \text{iff} \quad \lim_{R \nearrow 1} \|R^{\perp}TR\| = 0.$$

We remark that $\lim_{R \nearrow 1} \sup \|R^{\perp}TR\| = \varepsilon > 0$ means that for every $\eta < \varepsilon$ and every $R_0 \in$ lat$_f \mathfrak{D}(\mathcal{E})$ there exists $R \in$ lat$_f \mathfrak{D}(\mathcal{E})$ such that $R > R_0$ and $\|R^{\perp}TR\| > \eta$.

Proof. ⇒: Immediate.

⇐: We assume that $T \notin \mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$ and show that $\lim_{R \nearrow 1} \sup \|R^{\perp}TR\| > 0$. Note that the construction of Theorem 18 actually proves that $\mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$ is the intersection of all quasitriangular algebras $\mathfrak{Q}\mathfrak{T}(\{\Sigma_{\pi(i) < n} E_i\}_{n=1}^{\infty})$, where π is any permutation of N. Hence, if $T \notin \mathfrak{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H})$, then there is a permutation π of N and $\varepsilon > 0$ such that $\|(\Sigma_{\pi(i) < n} E_i)^{\perp} T (\Sigma_{\pi(i) < n} E_i)\| = \varepsilon$, and the reader can easily verify that $\lim_{R \nearrow 1} \sup \|R^{\perp}TR\| = \varepsilon$, thus concluding the argument. □

THEOREM 20. *The essential commutant of* $\mathfrak{Q}\mathfrak{D}(\mathcal{P})$ *is* $\mathbf{C} \cdot 1 + \mathcal{C}(\mathcal{H})$.

PROOF. To simplify the notation we again assume that dim $P_n = n$ for all n and remark that the most general case can be argued from this. Since $\mathfrak{D}(\mathcal{P}) + \mathcal{C}(\mathcal{H}) \subseteq \mathfrak{Q}\mathfrak{D}(\mathcal{P})$, it follows that the essential commutant of $\mathfrak{Q}\mathfrak{D}(\mathcal{P})$ is contained in the essential commutant of $\mathfrak{D}(\mathcal{P}) + \mathcal{C}(\mathcal{H})$ which by [8, Theorem 2.1] is $\mathfrak{D}(\mathcal{P})' + \mathcal{C}(\mathcal{H})$. If the essential commutant of $\mathfrak{Q}\mathfrak{D}(\mathcal{P})$ is larger than $\mathbf{C} \cdot 1 + \mathcal{C}(\mathcal{H})$, then it must contain a unitary operator U whose

Weyl spectrum has at least two distinct points λ_1 and λ_2. So, $U = V + K$, where K is compact and $V \in \mathfrak{D}(\mathfrak{P})'$ and by perturbing by another compact operator if necessary we insure that $V = \sum_{i=1}^{\infty} \mu_i E_i$, for $E_i = P_i \ominus P_{i-1}$, with λ_1 and λ_2 repeated infinitely often in the sequence $\{\mu_i\}_{i=1}^{\infty}$. Therefore, there is a subsequence $\{i_k\}_{k=1}^{\infty}$ of $\mathbf{N}$ such that

$$V = \sum_{k \text{ even}} \lambda_1 E_{i_k} \oplus \sum_{k \text{ odd}} \lambda_2 E_{i_k} \oplus W.$$

By Lemma 17 there is a noncompact operator

$$T \in \mathfrak{Q}\mathfrak{D}\left(\left\{\sum_{\substack{k=1 \\ k \text{ odd}}}^{n} E_{i_k}\right\}_{n=1}^{\infty}\right) \quad \text{with } \delta(T) = 0.$$

Let X be a partial isometry with initial space $(\sum_{k \text{ odd}} E_{i_k})\mathcal{H}$ and final space $(\sum_{k \text{ even}} E_{i_k})\mathcal{H}$ such that $XE_{i_k}X^* = E_{i_{k+1}}$ for each odd k; note that $XT \in \mathfrak{Q}\mathfrak{D}(\mathfrak{P})$ and is not compact so that $V(XT) - (XT)V = (\lambda_1 - \lambda_2)XT \notin \mathcal{C}(\mathcal{H})$. Because of this contradiction, we conclude that the essential commutant of $\mathfrak{Q}\mathfrak{D}(\mathfrak{P})$ is $\mathbf{C} \cdot 1 + \mathcal{C}(\mathcal{H})$. $\square$

We remark that Theorem 20 is particularly interesting, as recently, in [11, Corollary 1.9], it was shown that every unital, separable, C^*-subalgebra of the Calkin algebra equals its double commutant; $\pi(\mathfrak{Q}\mathfrak{D}(\mathfrak{P}))$ is the only example known to the author of a unital C^*-subalgebra of the Calkin algebra which does not equal its double commutant.

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